

NOTE ON LÖWNER'S THEOREM ON MATRIX MONOTONE FUNCTIONS IN SEVERAL COMMUTING VARIABLES OF AGLER, MCCARTHY AND YOUNG

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ABSTRACT. In this brief note, we show that the hypotheses of Löwner's theorem on matrix monotonicity in several commuting variables as proved by Agler, McCarthy and Young can be significantly relaxed. Specifically, we extend their theorem from continuously differentiable locally matrix monotone functions to arbitrary locally matrix monotone functions using mollification techniques.

1. INTRODUCTION

A function $f : (a, b) \rightarrow \mathbb{R}$ is *matrix monotone* if

$$A \leq B \Rightarrow f(A) \leq f(B)$$

for all A, B self adjoint matrices with spectrum in (a, b) , where $A \leq B$ means $B - A$ is positive semidefinite. In 1934[3], Charles Löwner showed that if $f : (a, b) \rightarrow \mathbb{R}$ is matrix monotone, then f analytically continues to the upper half plane $\Pi \subset \mathbb{C}$ as a map $f : \Pi \cup (a, b) \rightarrow \overline{\Pi}$.

Agler, McCarthy and Young extended Löwner's theorem to several commuting variables for the class of locally matrix monotone functions. Let E be an open subset of \mathbb{R}^d . Let $CSAM_n^d(E)$ denote the d -tuples of commuting self-adjoint matrices of size n with joint spectrum contained in E . (That is, if you jointly diagonalize an element of $CSAM_n^d(E)$ it should look like a direct sum of elements of E .) A *locally matrix monotone function* is a function $f : E \rightarrow \mathbb{R}$ so that for every n , on every C^1 curve $\gamma : [0, 1] \rightarrow CSAM_n^d(E)$ such that $\gamma'(t)_i \geq 0$ for all i and all $t \in [0, 1]$,

$$t_1 \leq t_2 \Rightarrow f(\gamma(t_1)) \leq f(\gamma(t_2)). \quad (1.1)$$

We generalize the following result.

Theorem 1.1 (Agler, McCarthy, Young [1]). *Let E be an open subset of \mathbb{R}^d . A C^1 function $f : E \rightarrow \mathbb{R}$ is locally matrix monotone if and*

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only if f is analytic and f analytically continues to Π^d as a map $f : E \cup \Pi^d \rightarrow \overline{\Pi}$.

We show that the assumption that f is C^1 can be dropped as in Löwner's theorem for one variable. That is, we prove the following theorem.

Theorem 1.2. *Let E be an open subset of \mathbb{R}^d . Let $f : E \rightarrow \mathbb{R}$ be a locally matrix monotone function, then f is analytic.*

So as an immediate corollary, we obtain the following.

Corollary 1.3. *Let E be an open subset of \mathbb{R}^d . A function $f : E \rightarrow \mathbb{R}$ is locally matrix monotone if and only if f is analytic and f analytically continues to Π^d as a map $f : E \cup \Pi^d \rightarrow \overline{\Pi}$.*

2. PROOF OF THE RESULT

We fix the convention that for $a, b \in \mathbb{R}^d$, $a < b$ means that $b_i - a_i > 0$ for all i and $a \leq b$ means that $b_i - a_i \geq 0$. Furthermore for $a \in \mathbb{R}^d$ we define $\|a\| = \|a\|_\infty = \sup_i |a_i|$. With the ordering induced by \leq , locally matrix monotone functions define on a ball with respect to the above norm are monotone in the more conventional sense that

$$a \leq b \Rightarrow f(a) \leq f(b). \quad (2.1)$$

To prove Theorem 1.2 we mollify a locally matrix monotone function f to get a smooth matrix monotone function and use the analytic continuations of those from Theorem 1.1 to derive an analytic continuation for f itself.

Let E be an open subset of \mathbb{R}^d . Fix $f : E \rightarrow \mathbb{R}$ be a locally matrix monotone function. Fix K an open set such that \overline{K} is compact and $\overline{K} \subset E$. Note that it is sufficient to prove that f is analytic on each such K since analyticity is a local property. Let μ be Lebesgue measure. Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a nonnegative smooth function with compact support so that $\int_{\mathbb{R}^d} \phi(x) d\mu(x) = 1$. Let $\phi_t(x) = t^{-d} \phi(t^{-1}x)$. Let $f_t = \phi_t * f$ where f is formally extended to be 0 off E . Let $\vec{1} = (1, 1, \dots, 1) \in \mathbb{R}^d$.

We note f_t is well-defined since f is continuous, because, for any $x \in E$, the function $f(x + \epsilon \vec{1})$ is continuous as a function of ϵ by the classical Löwner theorem and, for all $y \in E$ such that $\|x - y\| < \epsilon$,

$$f(x - \epsilon \vec{1}) \leq f(y) \leq f(x + \epsilon \vec{1}).$$

Furthermore, we note $f_t \rightarrow f$ pointwise since f is continuous.

Now, we give criteria for the mollification f_t of a locally matrix monotone function to be locally matrix monotone.

Claim 2.1. *The function f_t is locally matrix monotone on K for sufficiently small t .*

Proof. Let t be small enough so that $K + \text{Supp } \phi_t \subset E$. If $a \leq b$,

$$f_t(\gamma(b)) - f_t(\gamma(a)) = \int_{\text{Supp } \phi_t} \phi_t(\xi)(f(\gamma(b) - \xi) - f(\gamma(a) - \xi))d\mu(\xi) \geq 0$$

since

$$f(\gamma(b) - \xi) - f(\gamma(a) - \xi) \geq 0$$

because $\gamma_\xi = \gamma - \xi$ is itself a path in $CSAM_n^d(E)$ such that $\gamma'_\xi(x) \geq 0$ and so by the definition of local matrix monotonicity, we are done. \square

To show that the f_t have an analytic limit as $t \rightarrow 0$ we will show that they form a normal family.

Theorem 2.2 (P. [4, Theorem 4.4]). *Let $F \subset \mathbb{R}^d$ be open. Let $p \in F$. There are absolute constants q_k^p so that q_k^p is $O(c^k)$ for some $c > 0$ so that for every differentiable locally matrix monotone function $f : F \rightarrow \mathbb{R}$,*

$$\frac{|f^{(k)}(p)[\vec{z}]|}{k!} \leq q_k^p \|\vec{z}\|^k f'(p)[\vec{1}].$$

(Here $f^{(k)}(p)[\vec{z}] = \frac{d^k}{dt^k} f(p + t\vec{z})$.)

Namely, if $F = \{a \mid \|a - x\| \leq \delta\}$, there are constants c and D such that for every differentiable locally matrix monotone function $f : F \rightarrow \mathbb{R}$, f analytically continues to all $z \in \mathbb{C}^d$ such that $\|z - x\| \leq c/\delta$ and

$$|f(z) - f(x)| \leq D \frac{\|z\|}{1 - \frac{c}{\delta}\|z\|} f'(x)[\vec{1}].$$

Theorem 2.2 implies $(f_t)_{0 < t < \epsilon}$ form a normal family if for any basic open set in the domain of f we can find an a_0 such that $\limsup_{t \rightarrow 0} f'_t(a_0)[\vec{1}] < \infty$.

Claim 2.3. *Let $x \in K$. For any $\delta > 0$, there is an $a_0 \in K$ such that $\|x - a_0\| < \delta$ and $\limsup_{t \rightarrow 0} f'_t(a_0)[\vec{1}] < \infty$.*

Proof. Let $\hat{K} = \{a \mid \|a - x\| \leq \delta\}$. Note it is sufficient to prove the claim for all sufficiently small values of δ so that $\hat{K} \subset K$. Let ρ be the weak limit of the measures

$$\rho_\epsilon(x) = \frac{f(x + \epsilon\vec{1}) - f(x)}{\epsilon} d\mu$$

as $\epsilon \rightarrow 0$ taken in the dual of $C_0(\hat{K})$ which exists because f is locally bounded since it is monotone. Thus, the total variation is bounded as

follows:

$$\begin{aligned}
|\rho_\epsilon| &= \frac{1}{\epsilon} \left[\int_{\hat{K} + \epsilon \vec{1}} f d\mu - \int_{\hat{K}} f d\mu \right] \\
&\leq \frac{1}{\epsilon} \int_{(\hat{K} + \epsilon \vec{1}) \setminus \hat{K} \cup \hat{K} \setminus (\hat{K} + \epsilon \vec{1})} |f| d\mu \\
&\leq \frac{\sup_{(\hat{K} + \epsilon \vec{1}) \setminus \hat{K} \cup \hat{K} \setminus (\hat{K} + \epsilon \vec{1})} |f| \mu((\hat{K} + \epsilon \vec{1}) \setminus \hat{K} \cup \hat{K} \setminus (\hat{K} + \epsilon \vec{1}))}{\epsilon} \\
&\leq \frac{\max(|f(x + (\epsilon + \delta) \vec{1})|, |f(x - \delta \vec{1})|)}{\epsilon} 2d\delta^{d-1}\epsilon \\
&= \max(|f(x + (\epsilon + \delta) \vec{1})|, |f(x - \delta \vec{1})|) 2d\delta^{d-1}.
\end{aligned}$$

Note $f'_t(x)[\vec{1}] = \phi_t * \rho$. Now, pick a point a_0 in the interior of \hat{K} so that the density of ρ with respect to Lebesgue measure is finite. (Such a point exists by [2, pg 99, Theorem 3.22]) Note that $\limsup_{t \rightarrow 0} f'_t(a_0)[\vec{1}]$ is equal to the density of ρ with respect to Lebesgue measure, so we are done. \square

Thus, f is analytic.

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